Inertial Oscillations in Geostrophic Flow: Is the Inertial Frequency Shifted by $\zeta/2$ or by ζ?

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ABSTRACT

The short answer to the question posed in the title is that it depends on the frame of reference chosen to describe the motions. In the inertial limit, the frequency in a rotating frame of reference corresponds to the rotation rate of the inertial current vectors relative to that frame. When described in a reference frame rotating with a geostrophic flow having a relative vertical vorticity ζ, inertial oscillations have a frequency $f + \zeta$, equal to twice the fluid's rotation rate around the local vertical axis. From a nonrotating frame of reference, one would measure only half this frequency; the other half arises from describing inertial motions in a reference frame rotating with the background flow. However, when described in a reference frame rotating with Earth, hence rotating at $-\zeta/2$ relative to the geostrophic frame, inertial oscillations have a frequency reduced to $f + \zeta/2$.

1. Introduction

In his often-cited paper on the effects of geostrophic flow on near-inertial internal wave propagation, Kunze (1985, hereafter K85) showed that the geostrophic vorticity ζ shifts the lower bound of the internal wave band from the inertial frequency f to an effective inertial frequency $f_{\text{eff}} = f + \zeta/2$, a result established earlier, albeit with more restrictive assumptions (Fomin 1973; Mooers 1975; Weller 1982). The intuitive explanation for this result is that "a wave experiences the fluid's as well as Earth's rotation" (K85). However, because Earth's rotation rate Ω gives rise to inertial oscillations with frequency $f = 2\Omega \sin\theta$ (where θ is latitude), which is twice the rotation rate around the local vertical direction, one could naively expect geostrophic vorticity to shift the frequency by ζ, which is twice the rotation rate of fluid parcels relative to Earth. K85 argued that the $\zeta/2$ shift results from the use of rectilinear instead of polar or spherical coordinates in his treatment of the problem.

Kunze et al. (1995, hereafter K95) and Kunze and Boss (1998, hereafter KB98) used polar coordinates to treat the somewhat different problem of near-inertial waves trapped within the core of an axisymmetric vortex and found that the lower bound of the trapped internal wave band was shifted to $f + \zeta$, seemingly confirming the explanation given by K85.

We find this explanation unsatisfactory, because the frequency shift of near-inertial waves in geostrophic flow should not depend on the type of coordinates used to solve the problem. The purpose of this note is to demonstrate that the fundamental difference between the results obtained by K85, K95, and KB98 arises in fact from the different rotation rates of the frames of reference chosen to describe the motions. Durran (1993) showed that, although inertial oscillations on the North Pole f plane have a frequency f when described in a reference frame rotating with Earth, they have a frequency $f/2$ when described in an inertial (nonrotating) reference frame. We extend in section 2 the approach of Durran (1993) to include a solid-body rotating vortex centered on the North Pole. This provides the simplest conceptual framework to understand the fundamental difference between the results obtained by K85, K95, and KB98, as discussed in section 3. Although it is more

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difficult to visualize inertial oscillations in alternative reference frames for an *f* plane at an arbitrary latitude than for the polar plane, there is no difference in the physics or the mathematics. An extension to arbitrary background flows is given in the appendix.

2. Inertial oscillations in solid-body rotating flow on the polar *f* plane

Let us investigate inertial oscillations on the North Pole *f* plane, in the presence of a mean solid-body rotating vortex centered on the North Pole, with constant relative vertical vorticity ζ. We use three different frames of reference to describe the motions: (i) an inertial (i.e., nonrotating) frame, (ii) a frame rotating with the vortex flow, and (iii) a frame rotating with Earth.

a. Inertial reference frame

A fluid parcel's motion in an inertial reference frame I is governed by Newton's second law, which, for a frictionless fluid, takes the form

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathrm{I}} = \mathbf{F}_g + \mathbf{F}_p, \tag{1}$$

where $(d/dt)_{\mathrm{I}}$ is the material derivative in I (following the fluid parcel's motion relative to I), $\mathbf{x}$ is the position of the fluid parcel relative to the North Pole, $\mathbf{F}_g$ is the gravitational force per unit mass, $\mathbf{F}_p = -\rho_0^{-1}\nabla p$ is the pressure gradient force per unit mass, ρ_0 is density (assumed constant here), and p is pressure (e.g., Gill 1982, p. 72).

In the absence of any motion relative to Earth other than the solid-body rotating flow, the fluid is in counterclockwise solid-body rotation about the North Pole, with an absolute angular velocity $\boldsymbol{\Omega}_a = [(f + \zeta)/2]\mathbf{k}$, where $f = 2\Omega$ is the Coriolis parameter at 90°N (assumed greater than $|\zeta|$) and $\mathbf{k}$ is the vertical unit vector. The gravitational and pressure gradient forces therefore combine to yield the required centripetal acceleration,

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathrm{I}} = -\left(\frac{f+\zeta}{2}\right)^2 \mathbf{x}. \tag{2}$$

The free surface of the fluid is warped relative to the geoid, causing the familiar horizontal (i.e., along the geoid) pressure gradients responsible for geostrophic flows and allowing fluid parcels to rotate at a rate different from Earth's (Fig. 1).

Suppose the circular motion of a fluid parcel around the North Pole described above is momentarily disrupted (e.g., by a sporadic wind event for a surface water parcel) but the force having caused the disruption subsequently vanishes. Assume the impulsive forcing is of
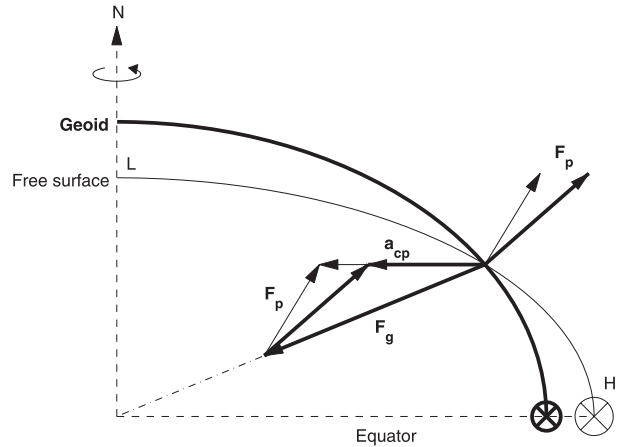

FIG. 1. Balance of forces for a frictionless homogeneous fluid entirely covering the smooth surface of a hypothetical Earth: if the fluid is at rest with respect to Earth (thick curve and arrows), the free surface corresponds to the geoid (shown with greatly exaggerated eccentricity), whereas, if the fluid rotates more rapidly than Earth (thin curve and arrows)—that is, in the presence of a cyclonic vortex in solid-body rotation centered on the North Pole—the free surface is warped, causing pressure to decrease at the North Pole (indicated by L) and to increase at the equator (indicated by H). The fluid velocity increases from zero at the North Pole to a maximum value at the equator, indicated by the arrows into the page. The centripetal acceleration $\mathbf{a}_{\mathrm{cp}}$ equals the sum of the gravitational force $\mathbf{F}_g$ and the pressure gradient force $\mathbf{F}_p$. For simplicity, $\mathbf{F}_g$ is shown pointing toward the center of Earth, although in reality it is slightly deviated by the equatorial bulge. The balance of forces is shown at a latitude far from 90°N for clarity, with the differences between the thick and thin arrows getting smaller as the North Pole is approached but remaining qualitatively similar.

large horizontal extent, so that subsequent horizontal pressure and velocity gradients are negligible. After the disruption, the fluid parcel's inertial acceleration is again given by (2), and the new motion of the fluid parcel is a counterclockwise ellipse with frequency $(f + \zeta)/2$. This is the motion of inertial oscillations in solid-body rotating flow, as seen by an observer fixed in I.

b. Reference frame rotating with the vortex flow

A natural frame of reference to describe inertial oscillations in the presence of a mean solid-body rotating flow is a frame F rotating with the mean flow at a constant angular velocity $\boldsymbol{\Omega}_a$ relative to I. Given any two frames of reference, A and B, where the angular velocity of B relative to A is $\boldsymbol{\omega}$, the rates of change of any vector $\mathbf{Q}$ in A and B are related by

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathrm{A}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathrm{B}} + \boldsymbol{\omega} \times \mathbf{Q}. \tag{3}$$

Letting A be the inertial frame I and B the rotating frame F and applying (3) first to the position vector, $\mathbf{Q} = \mathbf{x}$,

and then to the velocity vector in I, $\mathbf{Q} = (d\mathbf{x}/dt)_I$, yields the familiar expression

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_I = \left(\frac{d^2\mathbf{x}}{dt^2}\right)_F + 2\mathbf{\Omega}_a \times \mathbf{V}_F + \mathbf{\Omega}_a \times (\mathbf{\Omega}_a \times \mathbf{x}), \quad (4)$$

where $\mathbf{V}_F = (d\mathbf{x}/dt)_F$ is the velocity of inertial oscillations in F. Combining (4) and (2) yields

$$\left(\frac{d\mathbf{V}_F}{dt}\right)_F + (f + \zeta)\mathbf{k} \times \mathbf{V}_F = 0. \quad (5)$$

Here, $(d/dt)_F$ can be considered as the material derivative or the Eulerian derivative in F, because in the inertial limit the velocity field has no horizontal gradient.

Equation (5) is similar to that satisfied by inertial oscillations in a background flow at rest with respect to Earth, except that f is replaced by $f + \zeta$ here. The motions of fluid parcels are clockwise circles, with frequency $f + \zeta$ corresponding to the angular frequency of the velocity vector at a fixed point in F or following the fluid parcels. This is the motion of inertial oscillations in solid-body rotating flow, as seen by an observer fixed in F. The frequency measured in F is twice that measured in I and has the opposite sign (clockwise versus counterclockwise), because an observer fixed in F follows a circle, whereas a fluid parcel follows an ellipse, both at the same frequency relative to I. This is illustrated for $\zeta = 0$ in Fig. 2, in Durran (1993, Fig. 3), and in Stommel and Moore (1989, Fig. 3.1).

c. Reference frame rotating with Earth

Because we observe flows relative to Earth, we usually describe inertial oscillations in a reference frame E, rotating with Earth. At a given location in E, the velocity is the vortex flow, which is a function of position but not of time, plus $\mathbf{V}_F$, which is a function of time but not of position. Therefore, the measured inertial oscillation is still just $\mathbf{V}_F$ but viewed from the new reference frame E. As E rotates at $-\zeta/2$ relative to F, the application of (3) with $\mathbf{Q} = \mathbf{V}_F$ yields

$$\left(\frac{d\mathbf{V}_F}{dt}\right)_F = \left(\frac{d\mathbf{V}_F}{dt}\right)_E - \frac{\zeta}{2}\mathbf{k} \times \mathbf{V}_F. \quad (6)$$

Combining (6) and (5) yields

$$\left(\frac{d\mathbf{V}_F}{dt}\right)_E + \left(f + \frac{\zeta}{2}\right)\mathbf{k} \times \mathbf{V}_F = 0. \quad (7)$$

Here again, $(d/dt)_E$ can be considered in the inertial limit as the material derivative or the Eulerian derivative in E. The trajectories of fluid parcels are clockwise circles,
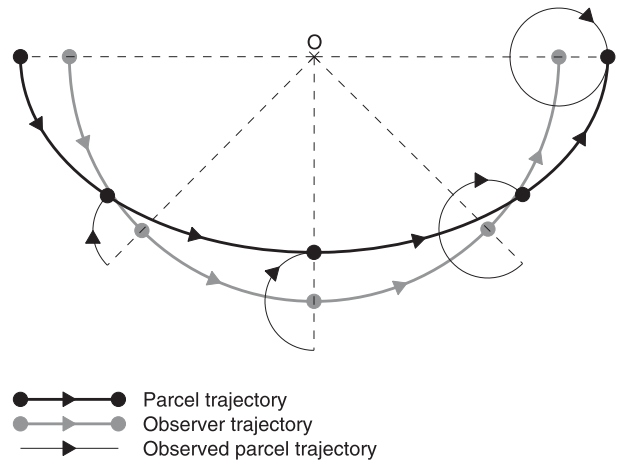

FIG. 2. An illustration of inertial oscillation on the North Pole (labeled O) f plane. The thick black curve and bullets represent the elliptical counterclockwise trajectory and the positions of a fluid parcel undergoing inertial oscillation at particular times during half a day, as seen by an observer fixed in an inertial (nonrotating) reference frame. To this observer, the rotation rate of the fluid parcel is equal to Earth's rotation rate. The thick gray curve and bullets represent the circular counterclockwise trajectory and the positions of an observer fixed in the rotating f plane at the same particular times. To this observer, the fluid parcel's motion appears as a circular clockwise trajectory (thin black curves), with twice Earth's rotation rate: the fluid parcel completes a full circle around the rotating observer during half a day.

with frequency $f + \zeta/2$ corresponding to the angular frequency of the velocity vector at a fixed point in E or following the fluid parcels, superimposed on the mean vortex flow. This is the motion of inertial oscillations in the presence of solid-body rotating flow, as seen by an observer fixed in E. The frequency measured in E is $\zeta/2$ less than that measured in F, because E rotates at $-\zeta/2$ relative to F, and the frequencies correspond to the angular frequencies of the velocity vector relative to the rotating frames of reference.

3. Discussion

The main result of the previous section is that the trajectory and frequency of inertial motions depend on the rotation rate of the frame of reference chosen to describe them. It does not depend on the type of coordinates (e.g., Cartesian versus polar), as shown by our exclusive use of vectorial notation in section 2. There are three key points to understand the different frequencies measured in the three different frames of reference considered. First, inertial motions on the polar f plane in the presence of a solid-body rotating vortex centered on the North Pole are counterclockwise ellipses with frequency $(f + \zeta)/2$ in an inertial reference frame, due to the warping (relative to the geoid) of the free surface

associated with the vortex flow, which causes fluid parcels to rotate at a rate different from Earth's (Fig. 1). This is the fluid's rotation "felt" by the inertial motions. Second, inertial motions appear as clockwise circles with frequency $f + \zeta$ in a reference frame rotating with the vortex flow at angular velocity $(f + \zeta)/2$, because a rotating observer follows a counterclockwise circle at the same frequency as the fluid parcel's relative to I. This is analogous to the case of inertial oscillations in a background flow at rest with respect to Earth (Fig. 2). Third, noting that, in the inertial limit, the frequency in the rotating reference frame corresponds to the rotation rate of the inertial current vectors relative to that frame [Eq. (5)], it is straightforward to obtain a frequency of $f + \zeta/2$ in a reference frame rotating with Earth at $-\zeta/2$ relative to the reference frame rotating with the vortex flow.

Following Durran (1993), we restricted the theoretical treatment of section 2 to the North Pole f plane for simplicity. The results remain unchanged for f planes at any other latitude but are more difficult to visualize. The reference frame with zero angular velocity component in the local vertical direction is not an inertial frame, because it has a northward (i.e., horizontal) angular velocity component of $\Omega \cos\theta$. Insofar as all motions remain horizontal, however, this horizontal component of angular velocity has no effect. Restricting the discussion to f planes also neglects the important effects of the meridional variations of f on the propagation of near-inertial oscillations (e.g., Garrett 2001), which are beyond the scope of this short note. Indeed, we have restrained for simplicity our treatment to the inertial limit, in which inertial oscillations do not propagate.

The case of a solid-body rotating vortex provides the simplest conceptual framework to investigate the effect of background flow on inertial oscillations. This is a very particular case, however, because the strain rate of the flow is zero. Inertial oscillations in an arbitrary background flow have their frequency affected by both the background vorticity and strain rate (see appendix): the effect of the background vorticity on the inertial frequency is $\mathcal{O}(\mathrm{Ro})$, whereas the effect of the background strain rate is $\mathcal{O}(\mathrm{Ro}^2)$, where Ro is the Rossby number of the background flow. Therefore, the results of section 2 remain approximately valid for arbitrary geostrophic flows (Ro $\ll 1$).

K95 and KB98 described near-inertial motions trapped within the solid-body rotating core of an axisymmetric vortex and obtained an effective inertial frequency of $f + \zeta$ "in a mean Lagrangian frame following the core's rotation" (K95)—that is, a reference frame rotating with the vortex flow—consistent with the results of section 2b. In contrast, K85 described near-inertial motions in rectilinear geostrophic flow and obtained an effective inertial frequency of $f + \zeta/2$ in a mean Lagrangian frame translating, but not rotating, with the geostrophic flow—that is, a reference frame rotating with Earth only—consistent with the results of section 2c. This consistency is expected because the inertial limit we have considered here is a nonsingular limit of the more general internal wave propagation problem considered by K85, K95, and KB98. When observations are analyzed in a reference frame rotating with Earth, K85's treatment is relevant. For example, Elipot et al. (2010) tracked surface drifters through geostrophic currents inferred from satellite altimetry in the global ocean and found a frequency shift of $\sim 0.4\zeta$, closer to K85's result than to K95's and KB98's.

APPENDIX

Inertial Oscillations in Arbitrary Background Flow

Inertial oscillations $\mathbf{u}$ in a mean background flow $\overline{\mathbf{u}}$ and in the Earth reference frame are governed by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \overline{\mathbf{u}} + f\mathbf{k} \times \mathbf{u} = 0 \qquad (A1)$$

in the linear (small amplitude) and inertial (horizontally uniform $\mathbf{u}$) limit. Let us solve the problem on the f plane using Cartesian coordinates. Defining the vorticity $\zeta = \partial\overline{v}/\partial x - \partial\overline{u}/\partial y$, the divergence $\delta = \partial\overline{u}/\partial x + \partial\overline{v}/\partial y$, the shear strain rate $\sigma_s = \partial\overline{v}/\partial x + \partial\overline{u}/\partial y$, and the normal strain rate $\sigma_n = \partial\overline{u}/\partial x - \partial\overline{v}/\partial y$, (A1) can be expressed as

$$\frac{\partial u}{\partial t} + \left(\frac{\delta + \sigma_n}{2}\right)u + \left[\frac{\sigma_s}{2} - \left(f + \frac{\zeta}{2}\right)\right]v = 0 \quad \text{and} \quad (A2)$$

$$\frac{\partial v}{\partial t} + \left[\frac{\sigma_s}{2} + \left(f + \frac{\zeta}{2}\right)\right]u + \left(\frac{\delta - \sigma_n}{2}\right)v = 0. \qquad (A3)$$

Looking for a solution of the form $\mathbf{u} = \mathbf{u}_0 \exp(i\omega t)$ yields the dispersion relation

$$\omega = i\frac{\delta}{2} \pm \sqrt{\left(f + \frac{\zeta}{2}\right)^2 - \frac{\sigma^2}{4}}, \qquad (A4)$$

where $\sigma = (\sigma_n^2 + \sigma_s^2)^{1/2}$ is the total strain rate. The solution is oscillatory when the strain rate is not too large, $\sigma^2 < 4(f + \zeta/2)^2$, and the frequency of inertial oscillations is affected by the background vorticity and strain rate, whereas the background divergence causes exponential growth or decay of the solution.

Let $(\zeta, \sigma) = (U/L)(\zeta^*, \sigma^*)$, where U and L are typical velocity and horizontal scales characterizing the background flow. The second term on the rhs of (A4) becomes

$$f_{\text{eff}} = f \sqrt{1 + \text{Ro}\, \zeta^* + \text{Ro}^2 \left(\frac{\zeta^{*2} - \sigma^{*2}}{4} \right)}, \quad \text{(A5)}$$

where $\text{Ro} = U/fL$ is the Rossby number of the background flow. The strain rate only affects the frequency to $\mathcal{O}(\text{Ro}^2)$, whereas the vorticity affects the frequency to $\mathcal{O}(\text{Ro})$. For geostrophic flows ($\text{Ro} \ll 1$), (A5) approximates to $f_{\text{eff}} = f + \zeta/2$. The expression for the effective inertial frequency given in (A5) differs from those typically given in the literature (e.g., Fomin 1973; Weller 1982; Kunze 1985; with the typographical correction given by Jones 2005) by emphasizing the fact that the inertial frequency is only affected by the background vorticity and strain rate. Fomin (1973) had the square of the background flow divergence explicitly appearing in his expression for the effective inertial frequency [his Eq. (15)] and concluded that "the divergence always lowers the frequency of the inertial velocity oscillations." It turns out that the $\mathcal{O}(\text{Ro}^2)$ terms in his expression can be rewritten as the so-called Okubo–Weiss

parameter appearing in (A5), which only involves vorticity and strain rate, not divergence.

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